Financial Mathematics in Pāṭīgaṇita and Gaṇitatilaka

K. N.Srikkanth[1]

Abstract

This article summarizes financial mathematics concepts found in the ancient Indian arithmetic texts *Pāṭīgaṇita* (PG) and *Gaṇitatilaka* (GT). It highlights how the growth of commerce influenced mathematical development, leading to specific methods for calculations involving principal, simple interest, commissions, and partnerships, grouped under *Miśrakavyavahāra*. Both texts provide similar rules and examples for problems like determining principal and interest when the final amount and time are known. A key concept discussed is *Ekapatrīkaraṇam*, a method to calculate the average interest rate and average tenure for a portfolio of different loans issued at varying rates and durations. This early form of portfolio analysis is compared to modern concepts like Macaulay duration for bonds. The texts also contain rules to determine the time required for capital to multiply by a certain factor at simple interest, analogous to the modern "Rule of 72" (though the latter uses compound interest). Furthermore, a specific problem in PG involving a lender staying in a borrower's house in lieu of interest showcases an understanding similar to present value or annuity concepts. The article concludes that PG and GT demonstrate sophisticated financial thinking, incorporating ideas like time value of money and portfolio management relevant even today.

Keywords: Pāṭīgaṇita; Gaṇitatilaka; Financial Mathematics.

Introduction

Pāṭīgaṇita (PG) and Gaṇitatilaka (GT) is a subject dealing with arithmetic. It deals with basic number systems, counting, and fundamental operations such as addition, subtraction, multiplication and division. Just like Astronomy, construction of Vedic altars, etc. fuelled the development of mathematics, development of commerce and economics played pivotal role in shaping arithmetic. In arithmetic texts, we find numerous problems dealing with barter trade, application of rule of three to various problems, principal-interest, profit sharing in partnerships, allegation problems, and so on.

The development of commerce and trade fuelled crafting ancient Indian mathematics in the direction of developing special tools and application for calculating interest, principal and time value of money. Certain advanced concepts such as Ekapatrīkaraṇa are also described which resemble today's concept of financial portfolios. For this article, I have relied on Pāṭīgaṇita of Śrīdhara and GaṇitaTilaka of Śrīpati for expounding the above concepts while considering PG as primary text.

2. Miśrakavyavahāra-treatment of various financial factors in a problem under *Miśrakavyavahāra*, the following topics are being dealt with by *PG*: *Mūladhanavṛddhidhanavyavahāra* (calculation involving principal and interest); Problems involving rental income and interest expense set off; Portfolio of bonds; *Suvarṇavyavahāra*(this is unique to PG); *Prakṣepaphala* (Rules for finding returns on investments in a partnership); *Krayavikraya* (purchase and sale); *Śeṣārghacchedaviṣaya* (Problems involving in linear equation); *Mandaśīghragati-melāpakālaḥ* (Meeting of two travellers with slow and fast motions); *Mārgapramāṇa* (problems involving travellers travelling at different speeds); *PatitasyaBhāṭaka* (Problems in remunerating a person who carries a dripping oil can); *Vyāvṛttapuruṣadāyagrahaṇavibhāgaḥ* (Pricing of theatrical performance when audience stay for unequal duartions during the show); *Vāpīpūraṇakālaḥ* (Problems involving finding the required time for filling up a well); *Antarbhāṭaka* (remunerating a person for carrying commodity in terms of the carried commodity itself); *Rasaprastara* (Combination of savours); Various problems involving finding unknown numbers; Rule of inversion.In this article, the common concepts mentioned by *PG* and *GT* are discussed.

Mūladhanavriddhidhana: Problems involving principal and interest

This involves finding simple interest and principal for the following problem:

Given that P_1 yields I_1 interest for T_1 period of time, find I_2 and P_2 when T_2 and P_2+I_2 is given.

To solve this, let r be the rate of simple interest rate which is results in both I_1 and I_2 after T_1 and T_2 maturity of principals. Then it follows that $I_1 = r P_1T_1$ and $I_2 = r P_2T_2$.

In other words since $r = \frac{I_1}{P_1T_1} = \frac{I_2}{P_2T_2}$; $I_2 + P_2 = \frac{I_1}{P_1T_1}P_2T_2 + P_2 = \left(\frac{I_1T_2+P_1T_1}{P_1T_1}\right)P_2$.

Hence $\frac{(I_2+P_2)(P_1T_1)}{I_1T_2+P_1T_1} = P_2$ and $I_2 = \left(I_2 + P_2\right) - \frac{(I_2+P_2)(P_1T_1)}{I_1T_2+P_1T_1} = \frac{(I_2+P_2)(I_1T_2)}{I_1T_2+P_1T_1}$.

PG's text related to the same is given below (v. 47):

निजकालेनाऽऽहन्यात्प्रमाणराशिंफलेनपरकालम्।

[1] Research Associate, Kuppuswami Sastri Research Institute, Email: srikkanthkn@gmail.com

तौस्वयुतिहृतौस्यातांमिश्रगुणौमूलवृद्धिधने॥

[Multiply the argument (*pramāṇarāśi*) by its time, and the other time by the fruit (*phala*); divide each of those (products) by their sum, and multiply by the amount (*i.e.,*capital plus interest). The results (thus obtained) give the capital (*mūla-dhana*) and the interest (*vrddhi-dhana*) respectively.]

GT (v. 111) reads almost similar:

निजकालहतंप्रमाणराशिंपरकालंफलताडितंचकुर्यात्।

निजयोगहृतौविमिश्रनिघ्नौभवतोमूलकलान्तरेक्रमेण॥

Here *nijakāla* is T_1 and *pramāṇarāśi* is P_1. Both these are to be multiplied as P_1T_1. *Phala* is I_1. This should be multipled with *parakāla* – T_2. Both should be multipled as I_1T_2.When the *miśradhana* ($I_2 + P_2$) is multiplied separately by P_1T_1 or I_1T_2 and divided by sum of these two quantities, the result would be principal (P_2) and interest (I_2)respectively.

The problem given by *PG* (v. 53) is as below:

सार्धस्यशतस्यफलंसपादमासेनरूपमध्यर्धम्।

मूलफलैक्यंषट्कृतिरर्धयुताऽर्धाष्टमैर्मासैः॥

"The interest on $100\frac{1}{2}$ for one month and a quarter being $1\frac{1}{2}$, a certain sum amounts to $36\frac{1}{2}$ in a period of $7\frac{1}{2}$ months. (Find the sum and the interest accrued thereon)".

Applying the rule

$$P_2 = \frac{36\frac{1}{2} \times 100\frac{1}{2} \times 1\frac{1}{4}}{1\frac{1}{2} \times 7\frac{1}{2} + 100\frac{1}{2} \times 1\frac{1}{4}} = 33\frac{1}{2};\ I_2 = \frac{36\frac{1}{2} \times 1\frac{1}{2} \times 7\frac{1}{2}}{1\frac{1}{2} \times 7\frac{1}{2} + 100\frac{1}{2} \times 1\frac{1}{4}} = 3\ .$$

Adding P_2 and I_2 we get maturity amount is 36 ½ which is required amount in the calculation. *GT* (v. 110 gives similar example:

पञ्चकेनशतेनाब्देफलमूलयुतिःशतम्।

चतुरूनंसखेदृष्टंकिंमूलंकिंकलान्तरम्॥

Extending the above rule to calculate commission etc.

PG (v.48) also gives a rule whereby along with interest, professional fee for accountant (*vṛtti)*, and fee for scribe who drafts the agreement (*lekhaka*) are also considered:

कालप्रमाणघातःपरकालहताःफलादयश्चैते।

स्वयुतिहृतामिश्रगुणाःभवन्तिमूलादयःक्रमशः॥

[Divide the product of the argument and its time as also the fruit *i.e.* interest, etc. as multiplied by the other time by their sum, and then multiply them by the mixed amount: then are obtained the capital etc. in their respective order.]

GT (v. 113) gives the same rule as *PG* with slight modification in words to suit the metrical arrangement:

प्रमाणराशिर्निजकालनिघ्नोव्यतीतकालेनहतःफलादिः।

मिश्रस्वनिघ्नाविहृताःस्वयुत्यामूलादयस्तेक्रमशोभवन्ति॥

Example given by *PG* (v. 54) is as follows:

मासेनशतस्यफलंपञ्चैकोभाव्यकेऽर्धमथवृत्तौ।

लेखकपादौवर्षेपञ्चाधिकनवशतीमिश्रम्॥

[The rate of interest being 5 percent per month, the commission of the surety (*bhāvyaka)* 1 percent per month, the fee of the calculator (*vṛtti*) ½ per cent per month, and the charges of the scribe ¼ per cent per month, a certain sum amounts to 905 in a year. Find the capital, the interest, and the shares of the surety, calculator, and the scribe.]

Applying the rule mentioned above, the amounts are:

$$i)\ M\bar{u}ladhana = \frac{905}{100+12\times\left(5+1+\frac{1}{2}+\frac{1}{4}\right)} = 500;$$

$$ii)\ V\d{r}ddhi\ dhana = \frac{905\times 12\times 5}{100+12\times\left(5+1+\frac{1}{2}+\frac{1}{4}\right)} = 300;$$

$$iii)\ bh\bar{a}vyaka = \frac{905\times 12\times 1}{100+12\times\left(5+1+\frac{1}{2}+\frac{1}{4}\right)} = 60....$$

Similarly,

iv) vṛtti (30), and v) *lekhaka* (15) can also be arrived. The equation $\frac{(I_2+P_2)(P_1T_1)}{I_1T_2+P_1T_1}$ *is simplified as* $\frac{I_2+P_2}{1+\left(\frac{I_1T_2}{P_1T_1}\right)}$ above for calculating 4 and 5 extending it to commission etc and conveniently representing them within the text.

GT (v. 114) gives the same example in a more poetical manner in the *śārdhūlavikrīḍita* metric:

मासेनैकेनशतस्यकोविदफलंपञ्चैककोभाव्यकेवृत्तौद्रम्मदलंचलेखककृतेतद्वत्तुरीयांशकः।

मासैर्द्वादशभिःसखेनवशतीमिश्रंचपञ्चोत्तरंमूलाद्यंवदमिश्रकव्यवहृतौयद्यस्तितेकौशलम्॥

The commentary to *GT* (p. 84) introduces this as "*Vyājopajīvavṛttih*". Calculation of guarantee, commission for profession are only extensions of *Miśrakavyavahāra* rule as explained in *PG* above.

Relevance to modern financial mathematics

The calculation seems to be designed from borrower's perspective to give him the individual components of liability (in terms of commission, interest etc) given the overall total value of his liability. It is interesting to note the commission-based remuneration model existing in ancient days since the value of commission for scribe and accountant are calculated as a percentage of loan amount per month despite the amount of effort (drafting the agreement or undertaking accounting entries) being the same irrespective of amount of principal and tenure.

Ekapatrīkaraṇam – Finding average interest rate and tenure of different bonds

Here, the author discusses about averaging the duration and rate of return on different sums of principal lent at different interest rates for different periods (*PG*, v. 51):

गतकालफलसमासेमासफलैक्योद्धृतेभवेत्कालः।
शतगुणमासफलैक्येधनयोगहृतेशतस्यफलम्॥

[The sum (*samāsa*) of the interests (*phala*) (accruing on the given bonds) for the elapsed months (*gatakāla*), being divide by the sum (*aikya*) of the interests (on the same bonds) for one month, gives the time (in months for the equivalent single bond); and 100 times the sum of the interests for one month (on the bonds), being divided by the sum of the capitals (*dhanayoga*) (of the bonds), gives the rate of interest per cent (per month) (for the single bond).]

This rule seeks to find the average time and average interest rate applicable on a portfolio of bonds. GT (v. 120) gives similar rule in Mālinivṛtta:

गतसमयफलैक्येमासवृद्ध्यैकभक्तेभवतिहिगतकालोमासलाभैक्यभावे।
शतमपिचतस्मिन्ताडितेस्याच्छतेनद्रविणयुतिविभक्तेत्वेकपत्रीविधाने॥

To understand, let P_i be principal for which R_i is the interest percentage and T_i is the time frame for maturity. Then for question as to what the average interest rate should $\overline{R}$ and average time $\overline{T}$ such that $\sum_{i=1}^{i=n} \overline{PRT} = \sum_{i=1}^{i=n} (P_i R_i T_i)$.

As one of the solutions to the problem, under the above rule, *PG* requires one to calculate average interest rate using principals as weight and to calculate average time using principal and interest as weights.

That is $\overline{R} = \sum_{i=1}^{i=n} (P_i R_i) / \sum_{i=1}^{i=n} (P_i)$ and $\overline{T} = \sum_{i=1}^{i=n} (P_i R_i T_i) / \sum_{i=1}^{i=n} (P_i R_i)$ such that $(\sum_{i=1}^{i=n} P_i)\, \overline{RT} = \sum_{i=1}^{i=n} (P_i R_i T_i)$.

Example given by *PG* (vv. 57-8) is given below:

द्विकेत्रिकेचतुष्केचदत्तंस्वंपञ्चकेशते।
एकंद्वेत्रीणिचत्वारिशतान्येषांयथाक्रमम्॥
द्वौत्रयःपञ्चचत्वारोगतामासाद्विसङ्गुणा।
तत्कथंकथ्यतामेतैरेकपत्रंभविष्यति॥

[(There are 4 bonds on which) capitals amounting to 100, 200, 300 and 400 are given (to someone on interest) at the rates of 2, 3, 4 and 5 percent (per month) in the respective order; and months amounting to 2, 3, 5 and 4 each multiplied by 2, have passed (since the execution of the respective bonds). Say, how would a single bond (*eka-patra*) be now made out of these.]

GT (v. 123) gives similar example:

द्विकेत्रिकेचाथशतेचतुष्केयत्पञ्चकेधीरधनंप्रयुक्तम्।
सप्ताष्टषड्द्वादशतस्यमासाएकादिवृध्याक्रमशःशतानि॥

Solution:

Sl. No.	P	R	PR	T	PRT
1	100	2%	2	4	8
2	200	3%	6	6	36
3	300	4%	12	10	120
4	400	5%	20	8	160
Total	1000		40		324
Average		4%		8 $^{1}/_{10}$	

The total of 324 found in PRT column signifies total interest.

Applying the rule, the average interest rate is 4.00% and average time of maturity is 8 months and 3 days. A single bond of 1000 having 4.00% interest per month having a tenure of 8 months and 3 days would yield the same quantum of interest of 324. *GT* follows *PG* with respect to finding average interest rate and duration with respect to several sums lent out for different periods for different times. *GT* and *PG* treats average of time using principal and interest as weights, and average of interest rate using principal as weights.

Concept of Ekapatrīkaraṇam: Viewing portfolio of bonds as one 'average' bond

Both *PG* and *GT* give concept of finding average interest and time of portfolio of bonds called *ekapatrīkaraṇam.*Under this, a man lends different amounts at different interest rates for different tenures. The authors give the formula for arriving at average interest (using principals as weights) and average time (using principal and interest as weights). This analysis helps one to picture the overall earning-potential of combined portfolio by treating the total sum lent to different persons as one contract with average interest rate and average tenure.

Comparison with modern financial concept of duration of bond and interest rate sensitivity

In modern financial mathematics, bond fund managers use complex analytical tools such as 'Macaulay duration of bond' where the weighted average duration of regular coupon paying bond is found- *i.e.* – one regular coupon paying bond is seen as a mix of several zero-coupon paying bonds. This concept is further finetuned as 'modified duration of bond" that directly measures interest rate sensitivity of bonds. That is, when market yield goes higher, the market value of bond falls. The rate of change of bond's market price with respect to change in market yields is measured by modified duration of bonds.

The formula for finding Macaulay duration of bond[3] is $\sum_{t=1}^{t=n}\left(\frac{tC}{(1+y)^t}\right)+\frac{nF}{(1+y)^n}$ where C, the regular coupon, F is the face value (or maturity value), n is the tenure and y is the market determined yield given the risk profile of the the bond. This is of particular importance to fund-managers since Macaulay duration divided by (1+ yield) gives modified duration of bond which directly measures the sensitivity of bond prices to changes in market determined yield. The formula is:

$$\frac{1}{p}\left(\frac{dp}{dy}\right)=\left(\frac{-1}{1+y}\right)\left[\sum_{t=1}^{t=n}\left(\frac{tC}{(1+y)^t}\right)+\frac{nF}{(1+y)^n}\right] where\ p=\sum_{t=1}^{t=n}\left(\frac{C}{(1+y)^t}\right)+\frac{F}{(1+y)^n}\ and$$

$y = market\ yield$; which is equivalent to: $\frac{-Macaulay\ duration}{1+y} = Modified\ duration.$

In other words, the above relationship connects the weighted average tenure of bond to the level of price sensitivity towards market yields. Hence, in essence, a bond portfolio manager has to view one regular coupon paying bond as several zero-coupon bonds with multiple maturities to measure interest rate sensitivity in order to better undertake buy sell strategies for optimal earnings. The fact that *ekapatrīkaraṇa* works in much similar manner where several regular bonds lent at different interest rates for different periods is combined and viewed as one average synthetic bond - a concept which is precursor to modern day complex financial analysis – indicates the advancement of financial mathematics in ancient days.

Required tenure for money multiplication

This rule seeks to answer the question as to what should be the period of loan if *I*is the interest rate per period and Principal (*P*) should multiply x times. *PG* (v. 52ab) states: "कालप्रमाणघातःफलभक्तोव्येकगुणहतःकालः" (The product of the time and the argument, being divided by the fruit and (then) multiplied by the multiple minus *one*, gives the (required) time).

That is, if *I* is the interest rate per period, $P + P \times I \times n$ should be *xP*. Therefore *x*= 1+I_n. hence *n* = (*x* – 1)/*I*, and if *I* is expressed as $^r/_P$ then this equation is similar to rule laid above :- *n*= (*x* – 1) ($^P/_r$).

The example given by *PG* (v. 60ab) is below:

मासेन पञ्चकशते धनं प्रयुक्तं कदा भवेद्विगुणं ।

"A sum of money is put to interest at 5 percent per month. When will it become twice of itself"?

Applying the rule, *n*= (*x*-1) ($^P/_r$), i.e., (2 – 1) × (100/5) = 20 months. (or one year and 8 months). The author has given one more example.

GT (v. 117) :

कालगुणितं प्रमाणं व्येकगुणहतं कालः ।

follows *PG* in explaining the rule of finding the time required to multiply the capital lent out to a desired multiple given. It also gives an example (v. 118) similar to the one given by *PG*.:

शतद्वयस्य मासेन षड्द्रम्माः यदि वृद्धितः । त्रिगुणं केन कालेन प्रयुक्तं तद्धनं भवेत् ।।

From the definitions, sums and examples given above till this stage, it can be seen that, *GT* and *PG* closely follow each other except in few places highlighted above. Hereafter, starting with mixture of gold etc. *PG* has additional content which the extant *GT* does not have. Rules and examples have been given wherever necessary. Rules having similar subject have been aggregated and dealt together.

Relevance to Modern Concept: Doubling Time of Capital

In current financial world, the well-known rule 'Rule of 72' is often used to find out the doubling period of capital when interest is compounded and paid at maturity. It is common to ask as to 'how many times the capital would multiply in a given tenure'. It is also common to refer to inflation as "today's principal is worth only half its original value *n* years from now", which means that money's worth across time is understood in terms of convenient multiples of cash in hand at present. Rule of 72 is widely employed to find the doubling period of principal amount.

It seems that similar line of thinking existed among the businessmen in ancient days or even before which prompted them to separately devote a rule to answer such questions. The only difference between ancient and modern concept is that while the former deals with simple interest, the latter deals with compound interest.

Annuity Payment

PG has given the Rule and an example whereby the wealthy individual lends money to a land-lord (who is in need of money) and agrees to stay as tenant in his house in lieu of interest payment. The problem is to find number of months which the wealthy tenant needs to stay in borrower-landlord's house such that both are freed of their obligations. The calculation involved in solving the problem is similar to valuation of future cash flows using its present value.

Concept of Present Value

Present value is the current worth of a future sum of money or stream of cash flows (be it earnings or obligations) given a specified rate of return. For instance, 100 rupees invested in bank deposit which yields 8 percent interest every year would become 108 after one year (i.e., 100 = principal + 8 = interest or 100 × 1.08). It becomes 116.64 in the second year (100 = principal 16 = interest and 0.64 = interest on interest). One can therefore equate 108 which would accrue after one year to 100 rupees today. That is, the present value of 108 after one year is 100 today. Similarly present value of 116.64 two years from now is also worth 100 today. Erosion of purchasing power of money by inflation is also similarly denoted using present value. Modern financial mathematics adds any interest accrued on principal to the principal itself unless it is paid by the borrower regularly to the lender. The present value of single cash flow assumes that money is currently borrowed with an agreement to return it with all interest accrued thereon after certain number of years assuming no intermediate repayments or further borrowings. Any interest accrued thereon is added to the principal. Compounding of interest makes it easier to compare cash flows having irregular amounts and irregular intervals. For example, it helps in valuing large infrastructure projects which would have negative cash flows for initial few years but would have positive cash flows based on completion milestones. It is remarkable that *PG* has applied this concept to the 'swap arrangement' mentioned where lender-borrower agrees to lend money to borrower-owner in exchange for staying in latter's house. However, the key difference with current commercial practice is that the cash flows are discounted using present values under simple interest. Further, since rental obligations and interest obligations occur in future date, current commercial practice should be to appropriately net off against each other every month and only the balance should be considered for subsequent month. In case the outstanding balance is less than rental value per month, then one should find appropriate duration of month such that rental obligation proportional to the stay during the month is equal to outstanding principal and interest accrued thereof proportional to such duration.

Conclusion

Based on the above, both PG and GT have expounded many concepts of financial mathematics with examples that resemble today's approach of society towards the key concepts such as time value of money, considering multiple investments as a portfolio and so on.

Works Cited

Pāṭīgaṇita of Śrīdharcarya, with commentary ed. by Kripa Shankar Shukla, Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1959

Gaṇitatilaka of Śrīpati with commentary of Simhatilakasuri, Gaekwad's Oriental Series, Critically edited by H. R. Kapadia, Baroda Oriental Institute, 1937.